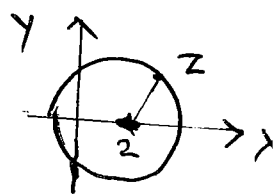


CALCULUS II: SAMPLE EXAM SOLUTIONS

1(a) $|z-2|^2 = |(x-2)+iy|^2 = (x-2)^2 + y^2 = 9$
 circle with centre (2,0) and radius 3
 $|z-2|$ is distance from (x,y) to (2,0)



(b) $16\sqrt{2}(1-i) = 32\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 32\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

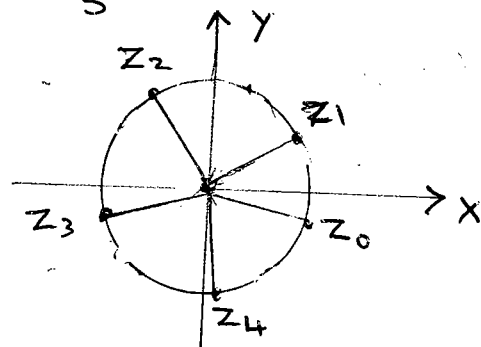
De Moivre $\Rightarrow$ one fifth root is $z_0 = 32^{\frac{1}{5}}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)$
 where $32^{\frac{1}{5}} = 2$ and $\pi/20 = 9^\circ$. Since roots are equally spaced around circle with separation $\frac{2\pi}{5} = 72^\circ$, others are

$$z_1 = 2 \operatorname{cis}\left(-\frac{\pi}{20} + \frac{2\pi}{5}\right) = 2 \operatorname{cis}\left(\frac{7\pi}{20}\right)$$

$$z_2 = 2 \operatorname{cis}\left(\frac{7\pi}{20} + \frac{2\pi}{5}\right) = 2 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

$$z_3 = 2 \operatorname{cis}\left(\frac{15\pi}{20} + \frac{2\pi}{5}\right) = 2 \operatorname{cis}\left(\frac{23\pi}{20}\right)$$

$$z_4 = 2 \operatorname{cis}\left(\frac{5\pi}{4} + \frac{2\pi}{5}\right) = 2 \operatorname{cis}\left(\frac{31\pi}{20}\right)$$



(c) $\sum_{n=1}^{\infty} \frac{10}{(5n-3)(5n+2)} = 2 \sum_{n=1}^{\infty} \left[\frac{1}{5n-3} - \frac{1}{5n+2} \right]$ (partial fractions)
 $= 2 \left\{ \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{7} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{17}\right) + \dots \right\} = 1.$

(d) $(1+x^3)^{-\frac{1}{4}} = 1 - \frac{1}{4}x^3 + \frac{(-\frac{1}{4})(-\frac{5}{4})}{2!}(x^6) + \frac{(-\frac{1}{4})(-\frac{5}{4})(-\frac{9}{4})}{3!}(x^9) + \dots$
 $= 1 - \frac{1}{4}x^3 + \frac{5}{32}x^6 - \frac{15}{128}x^9 + \dots$

(e) Ratio test: $\left| \frac{u_{n+1}}{u_n} \right| = \left(\frac{n}{n+1} \right) \frac{|x|}{4} \rightarrow \frac{|x|}{4} < 1$ for $|x| < 4$
 Hence converges for $-4 < x < 4$.

$x = 4 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n}$ (diverges); $x = -4 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ (converges)

f) $f = xe^y + y + 1 \Rightarrow f_x = e^y, f_y = xe^y + 1 \Rightarrow f_{xx} = 0,$
 $f_{xy} = e^y = f_{yx}, f_{yy} = xe^y$

$$(g) f = x^2 + 2xy - y^2 + z^2 - 7 \Rightarrow \nabla f = (2x+2y, 2x-2y, 2z)$$

$= (0, 4, 6)$ at $(1, -1, 3)$. Tangent plane is

$$(x-1)0 + (y+1)4 + (z-3)6 = 0 \Rightarrow 2y + 3z = 7$$

Normal is $x = 1, y = 4\lambda - 1, z = 6\lambda + 3$.

$$(h) \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$$

$$2. \text{L'Hôpital} \Rightarrow \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{x}{n}\right) = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{x}{n}\right)}{1/n} = \lim_{n \rightarrow \infty} \frac{-\frac{x}{n^2} \frac{1}{1+x/n}}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{x}{1+x/n} = x.$$

$$\text{Taking exponential: } \left(1 + \frac{x}{n}\right)^n \rightarrow e^x$$

$$\left(\frac{2n+1}{3n-1}\right)^n = \frac{\left(1 + \frac{1}{2n}\right)^n}{\left(1 - \frac{1}{3n}\right)^n} \left(\frac{2}{3}\right)^n \rightarrow \frac{e^{1/2}}{e^{-1/3}} \left(\frac{2}{3}\right)^n = e^{5/6} \left(\frac{2}{3}\right)^n$$

$$3. \text{Extremize } f = x - y + z \text{ such that } g = x^2 + y^2 + z^2 - 1 = 0$$

$$\nabla f = \lambda \nabla g \Rightarrow 1 = 2\lambda x, -1 = 2\lambda y, 1 = 2\lambda z$$

$$x^2 + y^2 + z^2 = 3 \left(\frac{1}{2\lambda}\right)^2 = 1 \Rightarrow \lambda = \pm \frac{\sqrt{3}}{2} \Rightarrow x = \pm \frac{1}{\sqrt{3}}, y = \mp \frac{1}{\sqrt{3}}, z = \pm \frac{1}{\sqrt{3}}$$

Max of $\sqrt{3}$ at $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, min of $-\sqrt{3}$ at $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$4. f = x^2 - 3xy + 5 \Rightarrow f_x = 2x - 3y, f_y = -3x$$

Linearization at $(2, 1)$ is $L(x, y) = (x-2)(1) + (y-1)(-6)$

$$= x - 6y + (4 - 6 + 5 - 2 + 6) = x - 6y + 7 \quad [+ f(2, 1)]$$

$$f_{xx} = 2, f_{xy} = -3, f_{yy} = 0 \Rightarrow |\max| = 3$$

$$|f(x, y) - L(x, y)|^2 = \frac{1}{2} \cdot 3 \left(|x-2| + |y-1| \right)^2 = 0.06$$

$$5. \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} dy dx = \int_0^1 \int_0^{3y^2} e^{y^3} dx dy$$

$$= \int_0^1 e^{y^3} [3y^2] dy = \int_0^1 e^{y^3} d(y^3) = [e^{y^3}]_0^1 = e - 1$$

